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IMPACT RESPONSE OF A TRANSVERSELY ISOTROPIC CYLINDER WITH A PENNY-SHAPED CRACK

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Abstract—The axisymmetric dynamic response of a penny-shaped crack in a transversely isotropic infinite cylinder under normal impact is analyzed. Laplace and Hankel transforms are used to reduce the transient problem to a pair of dual integral equations in the Laplace transform plane. The solution is given in terms of a Fredholm integral equation of the second kind. A numerical Laplace inversion routine is used to recover the time dependence of the solution. The dynamic stress intensity factor is determined and numerical results for some practical materials are shown graphically to demonstrate the influence of transverse isotropy.

1. INTRODUCTION

Dynamic fracture problems involving anisotropic materials weakened by crack-like imperfections have much attention because of the increased usage of macroscopically anisotropic construction materials such as fiber reinforced composites under impact or shock loading[1]. Kassir and Bandyopadhyay[2] considered the problem of an infinite orthotropic solid containing a central crack deformed by the action of suddenly applied stresses to its surfaces.

In this investigation, the normal impact response of a transversely isotropic cylinder with a penny-shaped crack is treated. The plane of the crack is perpendicular to the axis of the cylinder and is assumed to coincide with one of the planes of elastic symmetry of the material. Laplace and Hankel transforms are used to reduce the elastodynamic problem to a pair of dual integral equations in the Laplace transform plane. The solution is then given in terms of a Fredholm integral equation of the second kind having the kernel with finite integrals. A numerical Laplace inversion technique[3] is used to recover the time dependence of the solution. The dynamic stress intensity factor is computed and numerical values are shown in graphs for various transversely isotropic cylinders at designated time instances.

2. FORMULATION OF THE PROBLEM

Consider the axisymmetric problem of a transversely isotropic cylinder of radius b containing a penny-shaped crack of radius a and subjected to a time dependent applied stress as shown in Fig. 1. Let E_i, μ_{ij} and ν_{ij} $(i, j = 1, 2, 3)$ denote the engineering elastic

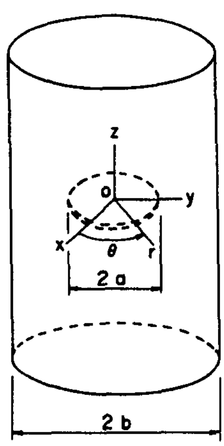

Fig. 1. Geometry of a transversely isotropic cylinder with a penny-shaped crack.

constants of the material where the subscripts 1, 2, 3 correspond to the directions (x, y, z) of a system of Cartesian coordinates chosen to coincide with the axis of material orthotropy. A cylindrical coordinate system (r, θ, z) is attached to the center of the crack that is symmetrically situated in the cylinder and the z-axis is parallel to the axis of symmetry of the transversely isotropic material. Let the components of the displacement vector $\mathbf{u}$ in the r, θ, z directions be labeled by u_r, u_θ and u_z. For axially symmetric deformation field, the nonzero stress components σ_r, σ_θ, σ_z and τ_{rz} are found as

$$\begin{aligned}
\frac{\sigma_r}{\mu_{13}} &= c_{11}\frac{\partial u_r}{\partial r} + c_{12}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} \\
\frac{\sigma_\theta}{\mu_{13}} &= c_{12}\frac{\partial u_r}{\partial r} + c_{11}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} \\
\frac{\sigma_z}{\mu_{13}} &= c_{13}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + c_{33}\frac{\partial u_z}{\partial z} \\
\frac{\tau_{rz}}{\mu_{13}} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}
\end{aligned} \tag{1}$$

where c_{ij} $(i, j = 1, 2, 3)$ are nondimensional parameters related to the elastic constants by the relations[4]:

$$\begin{aligned}
c_{11} &= \frac{E_1}{\Delta\mu_{13}}\left(1 - \frac{E_1}{E_3}\nu_{31}^2\right) \\
c_{33} &= \frac{E_3}{\Delta\mu_{13}}(1 - \nu_{12}^2) \\
c_{12} &= \frac{E_1}{\Delta\mu_{13}}\left(\nu_{12} + \frac{E_1}{E_3}\nu_{31}^2\right) \\
c_{13} &= \frac{E_1}{\Delta\mu_{13}}\nu_{31}(1 + \nu_{12}) \\
\Delta &= 1 - \nu_{12}^2 - 2\frac{E_1}{E_3}\nu_{31}^2(1 + \nu_{12}).
\end{aligned} \tag{2}$$

It is shown that the displacement equations of motion reduce to

$$\begin{aligned}
c_{11}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right] + \frac{\partial^2 u_r}{\partial z^2} + (1 + c_{13})\frac{\partial^2 u_z}{\partial r\,\partial z} &= \frac{1}{C_s^2}\frac{\partial^2 u_r}{\partial t^2} \\
c_{33}\frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + (1 + c_{13})\frac{1}{r}\frac{\partial^2}{\partial r\,\partial z}(ru_r) &= \frac{1}{C_s^2}\frac{\partial^2 u_z}{\partial t^2}
\end{aligned} \tag{3}$$

where t is the time and $C_s = (\mu_{13}/\rho)^{1/2}$ with ρ being the mass density of the material. In the isotropic solid, C_s represents the velocity of the shear wave.

Suppose that the penny-shaped crack is now loaded suddenly by a pair of normal stresses of magnitude $-\sigma_0$ such that the upper and lower crack surfaces move in the opposite directions. The surface of the cylinder is assumed to be stress free. The results can be used for a transient normal compression wave impinging on the flat penny-shaped crack and for the sudden appearance of a flat penny-shaped crack in a stressed medium under tension. Therefore, the boundary conditions may be written as

$$\tau_{rz}(r, 0, t) = 0 \qquad (0 \leqslant r \leqslant h) \tag{4}$$

$$\begin{aligned} &\sigma_z(r,0,t) = -\sigma_0 H(t) \qquad (0 \leqslant r < a) \\ &u_z(r,0,t) = 0 \qquad (a \leqslant r \leqslant b) \end{aligned} \tag{5}$$

$$\sigma_r(b,z,t) = 0 \tag{6}$$

$$\tau_{rz}(b,z,t) = 0 \tag{7}$$

where $H(t)$ is the Heaviside unit step function.

3. METHOD OF SOLUTION

Define a Laplace transform pair by the equations

$$f^*(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t, \qquad f(t) = \frac{1}{2\pi i}\int_{Br} f^*(p)\,e^{pt}\,\mathrm{d}p \tag{8}$$

in which Br stands for the Bromwich path of integration. The application of the first equation of (8) to equations (3) yields

$$\begin{aligned} &c_{11}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r^*)\right] + \frac{\partial^2 u_r^*}{\partial z^2} + (1+c_{13})\frac{\partial^2 u_z^*}{\partial r\,\partial z} - \frac{p^2}{C_s^2}u_r^* = 0 \\ &c_{33}\frac{\partial^2 u_z^*}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^*}{\partial r}\right) + (1+c_{13})\frac{1}{r}\frac{\partial^2}{\partial r\,\partial z}(ru_r^*) - \frac{p^2}{C_s^2}u_z^* = 0. \end{aligned} \tag{9}$$

By the use of an integral transform technique described in Appendix A, a proper solution to equations (9) can be obtained as

$$\begin{aligned} u_r^*(r,z,p) = \int_0^\infty &[\{A_1(s,p)e^{-\gamma_1 z} + A_2(s,p)e^{-\gamma_2 z}\}J_1(rs) \\ &+ \{A_3(s,p)I_1(\gamma_3 r) + A_4(s,p)I_1(\gamma_4 r)\}\cos(sz)]\,\mathrm{d}s \\ u_z^*(r,z,p) = \int_0^\infty &\left[\{\alpha_1 A_1(s,p)e^{-\gamma_1 z} + \alpha_2 A_2(s,p)e^{-\gamma_2 z}\}\frac{J_0(rs)}{s}\right. \\ &\left. + \{\alpha_3 A_3(s,p)I_0(\gamma_3 r) + \alpha_4 A_4(s,p)I_0(\gamma_4 r)\}\frac{\sin(sz)}{s}\right]\mathrm{d}s \end{aligned} \tag{10}$$

where A_1, A_2, A_3, A_4 are the unknowns to be solved, and $J_n(\ )$, $I_n(\ )$ are the Bessel functions of the first kind and the modified Bessel functions of the first kind of order n ($n = 0, 1$), respectively. γ_1^2, γ_2^2 and γ_3^2, γ_4^2 are the two roots of the quadratics:

$$c_{33}\gamma^4 + \left[(c_{13}^2 + 2c_{13} - c_{11}c_{33})s^2 - (1+c_{33})\frac{p^2}{C_s^2}\right]\gamma^2 + \left(s^2 + \frac{p^2}{C_s^2}\right)\left(c_{11}s^2 + \frac{p^2}{C_s^2}\right) = 0 \qquad (\gamma_1^2, \gamma_2^2) \tag{11}$$

$$c_{11}\gamma^4 + \left[(c_{13}^2 + 2c_{13} - c_{11}c_{33})s^2 - (1+c_{11})\frac{p^2}{C_s^2}\right]\gamma^2 + \left(s^2 + \frac{p^2}{C_s^2}\right)\left(c_{33}s^2 + \frac{p^2}{C_s^2}\right) = 0 \qquad (\gamma_3^2, \gamma_4^2). \tag{12}$$

Generally the roots γ_1, γ_2 and γ_3, γ_4 are complex. And, α_i $(i = 1\text{–}4)$ stand for the abbreviation

$$\alpha_i(s,p) = \begin{cases} \dfrac{c_{11}s^2 + \dfrac{p^2}{C_s^2} - \gamma_i^2}{(1+c_{13})\gamma_i} & (i = 1,2) \\[2ex] \dfrac{s^2 + \dfrac{p^2}{C_s^2} - c_{11}\gamma_i^2}{(1+c_{13})\gamma_i} & (i = 3,4). \end{cases} \tag{13}$$

The solution (10) would work for all materials. In the Laplace transform domain, equations (4)–(7) become

$$\tau_{rz}^*(r,0,p) = 0 \qquad (0 \leqslant r \leqslant b) \tag{14}$$

$$\sigma_z^*(r,0,p) = -\sigma_0/p \qquad (0 \leqslant r < a)$$

$$u_z^*(r,0,p) = 0 \qquad (a \leqslant r \leqslant b) \tag{15}$$

$$\sigma_r^*(b,z,p) = 0 \tag{16}$$

$$\tau_{rz}^*(b,z,p) = 0. \tag{17}$$

Substituting equations (10) into (1), one obtains the stress expressions in the Laplace transform plane. The satisfaction of equations (14) and (15) by these expressions yields

$$\int_0^\infty A(s,p)J_0(rs)\,\mathrm{d}s = 0 \qquad (a \leqslant r \leqslant b) \tag{18}$$

$$\int_0^\infty sF(s,p)J_0(rs)\,\mathrm{d}s = -\frac{\sigma_0}{p\theta_0\mu_{13}} - \frac{1}{\theta_0}\int_0^\infty [(c_{13}\gamma_3+c_{33}\alpha_3)A_3(s,p)I_0(\gamma_3 r) + (c_{13}\gamma_4+c_{33}\alpha_4)A_4(s,p)I_0(\gamma_4 r)]\,\mathrm{d}s \qquad (r > a) \tag{19}$$

where

$$A(s,p) = \frac{\alpha_1-\beta\alpha_2}{s}A_1(s,p) = \frac{\beta\alpha_2-\alpha_1}{s\beta}A_2(s,p) \tag{20}$$

$$F(s,p) = \frac{c_{13}s^2 - c_{33}\alpha_1\gamma_1 - \beta(c_{13}s^2 - c_{33}\alpha_2\gamma_2)}{(\alpha_1-\beta\alpha_2)s\theta_0} \tag{21}$$

$$\beta(s,p) = \frac{\alpha_1+\gamma_1}{\alpha_2+\gamma_2} \tag{22}$$

$$\theta_0 = \frac{(c_{13}^2+c_{13}-c_{11}c_{33})(c_{13}N_1N_2-c_{11}) - c_{33}\{c_{13}N_1^2N_2^2+c_{11}(N_1^2+N_1N_2+N_2^2)\}}{c_{11}(1+c_{13})(N_1+N_2)} \tag{23}$$

$$\begin{aligned} N_1^2 &= \frac{1}{2c_{33}}[(c_{11}c_{33}-c_{13}^2-2c_{13}) + \{(c_{11}c_{33}-c_{13}^2-2c_{13})^2-4c_{11}c_{33}\}^{1/2}] \\ N_2^2 &= \frac{1}{2c_{33}}[(c_{11}c_{33}-c_{13}^2-2c_{13}) - \{(c_{11}c_{33}-c_{13}^2-2c_{13})^2-4c_{11}c_{33}\}^{1/2}]. \end{aligned} \tag{24}$$

Through equations (16), (17), the unknowns $A_3(s,p)$ and $A_4(s,p)$ are related to the new parameter $A(s,p)$ by the following equations:

$$A_3(s,p) = \delta_1(s,p)\int_0^\infty g_1(s,\eta,p)A(\eta,p)\,d\eta + \delta_2(s,p)\int_0^\infty g_2(s,\eta,p)A(\eta,p)\,d\eta$$
$$A_4(s,p) = \delta_3(s,p)\int_0^\infty g_1(s,\eta,p)A(\eta,p)\,d\eta + \delta_4(s,p)\int_0^\infty g_2(s,\eta,p)A(\eta,p)\,d\eta \tag{25}$$

in which the functions $\delta_i(s,p)$ $(i = 1–4)$ and $g_i(s,\eta,p)$ $(i = 1,2)$ are

$$\delta_1(s,p) = -\frac{2}{\pi}\frac{1}{\Delta_s}\left(\frac{\alpha_4\gamma_4}{s} - s\right)I_1(\gamma_4 b)$$
$$\delta_2(s,p) = -\frac{2}{\pi}\frac{1}{\Delta_s}\left\{(c_{11}\gamma_4 + c_{13}\alpha_3)I_0(\gamma_4 b) + (c_{12}-c_{11})\frac{I_1(\gamma_4 b)}{b}\right\} \tag{26}$$
$$\delta_3(s,p) = \frac{2}{\pi}\frac{1}{\Delta_s}\left(\frac{\alpha_3\gamma_3}{s} - s\right)I_1(\gamma_3 b)$$
$$\delta_4(s,p) = \frac{2}{\pi}\frac{1}{\Delta_s}\left\{(c_{11}\gamma_3 + c_{13}\alpha_3)I_0(\gamma_3 b) + (c_{12}-c_{11})\frac{I_1(\gamma_3 b)}{b}\right\}$$

$$\Delta_s(s,p) = \left\{(c_{11}\gamma_3 + c_{13}\alpha_3)I_0(\gamma_3 b) + (c_{12}-c_{11})\frac{I_1(\gamma_3 b)}{b}\right\}\left(\frac{\alpha_4\gamma_4}{s} - s\right)I_1(\gamma_4 b)$$
$$-\left\{(c_{11}\gamma_4 + c_{13}\alpha_4)I_0(\gamma_4 b) + (c_{12}-c_{11})\frac{I_1(\gamma_4 b)}{b}\right\}\left(\frac{\alpha_3\gamma_3}{s} - s\right)I_1(\gamma_3 b) \tag{27}$$

$$g_1(s,\eta,p) = \frac{1}{\tilde{\alpha}_1 - \tilde{\beta}\tilde{\alpha}_2}\left[\frac{\tilde{\gamma}_1(c_{11}\eta^2 + (c_{12}-c_{11})\eta - c_{13}\tilde{\alpha}_1\tilde{\gamma}_1)}{s^2+\tilde{\gamma}_1^2}\right.$$
$$\left.-\tilde{\beta}\frac{\tilde{\gamma}_2(c_{11}\eta^2 + (c_{12}-c_{11})\eta - c_{13}\tilde{\alpha}_2\tilde{\gamma}_2)}{s^2+\tilde{\gamma}_2^2}\right]J_0(b\eta) \tag{28}$$
$$g_2(s,\eta,p) = \frac{s\eta(\tilde{\alpha}_1+\tilde{\gamma}_1)}{\tilde{\alpha}_1 - \tilde{\beta}\tilde{\alpha}_2}\left(\frac{1}{s^2+\tilde{\gamma}_1^2} - \frac{1}{s^2+\tilde{\gamma}_2^2}\right)J_1(b\eta)$$

$$\tilde{\alpha}_i(\eta,p) = \frac{c_{11}\eta^2 + \dfrac{p^2}{C_s^2} - \tilde{\gamma}_i^2}{(1+c_{13})\tilde{\gamma}_i}\ (i = 1,2), \qquad \tilde{\beta}(\eta,p) = \frac{\tilde{\alpha}_1+\tilde{\gamma}_1}{\tilde{\alpha}_2+\tilde{\gamma}_2}. \tag{29}$$

In equations (28) and (29), $\tilde{\gamma}_1^2$ and $\tilde{\gamma}_2^2$ are the two roots of the quadratic:

$$c_{33}\gamma^4 + \left[(c_{13}^2 + 2c_{13} - c_{11}c_{33})\eta^2 - (1+c_{33})\frac{p^2}{C_s^2}\right]\gamma^2 + \left(\eta^2 + \frac{p^2}{C_s^2}\right)\left(c_{11}\eta^2 + \frac{p^2}{C_s^2}\right) = 0. \tag{30}$$

The roots $\tilde{\gamma}_1^2$, $\tilde{\gamma}_2^2$ are subjected to the following relations:

$$\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 = -\frac{1}{c_{33}}\left[(c_{13}^2 + 2c_{13} - c_{11}c_{33})\eta^2 - (1+c_{33})\frac{p^2}{C_s^2}\right]$$
$$\tilde{\gamma}_1^2\tilde{\gamma}_2^2 = \frac{1}{c_{33}}\left(\eta^2 + \frac{p^2}{C_s^2}\right)\left(c_{11}\eta^2 + \frac{p^2}{C_s^2}\right)$$

$$(s^2+\tilde{\gamma}_1^2)(s^2+\tilde{\gamma}_2^2) = \frac{c_{11}}{c_{33}}(\eta^2+\gamma_3^2)(\eta^2+\gamma_4^2). \tag{31}$$

Using relations (31), we see that equations (28) are reduced to

$$
\begin{aligned}
g_1(s,\eta,p) &= \frac{1}{c_{11}}\left[-\frac{\gamma_3^2u_1-u_2}{\eta^2+\gamma_3^2}+\frac{\gamma_4^2u_1-u_2}{\eta^2+\gamma_4^2}\right]\frac{J_0(b\eta)}{\gamma_4^2-\gamma_3^2} \\
&\quad +\frac{\eta}{c_{11}}\left[-\frac{\gamma_3^2u_3-u_4}{\eta^2+\gamma_3^2}+\frac{\gamma_4^2u_3-u_4}{\eta^2+\gamma_4^2}\right]\frac{J_1(b\eta)}{\gamma_4^2-\gamma_3^2} \\
g_2(s,\eta,p) &= \frac{s\eta}{c_{11}}\left[-\frac{\gamma_3^2u_5-u_6}{\eta^2+\gamma_3^2}+\frac{\gamma_4^2u_5-u_6}{\eta^2+\gamma_4^2}\right]\frac{J_1(b\eta)}{\gamma_4^2-\gamma_3^2}
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
u_1 &= (c_{13}^2-c_{11}c_{33})s^2-c_{13}\frac{p^2}{C_s^2} \\
u_2 &= -c_{13}\left(s^2+\frac{p^2}{C_s^2}\right)\frac{p^2}{C_s^2} \\
u_3 &= \frac{1}{b}(c_{12}-c_{11})c_{13} \\
u_4 &= -\frac{1}{b}(c_{12}-c_{11})\left(c_{33}s^2-c_{13}\frac{p^2}{C_s^2}\right) \\
u_5 &= c_{11}c_{33}-c_{13}^2 \\
u_6 &= (c_{13}+c_{33})\frac{p^2}{C_s^2}.
\end{aligned} \tag{33}
$$

The dual integral equations (18) and (19) may be solved by using a procedure described in Ref. [5] and the result is

$$
A(s,p) = -\frac{2}{\pi}\frac{\sigma_0a^2}{\mu_{13}\theta_0p}\int_0^1 \Phi(\xi,p)\sin(sa\xi)\,\mathrm{d}\xi. \tag{34}
$$

In equation (34), the function $\Phi(\xi,p)$ is governed by the following Fredholm integral equation of the second kind:

$$
\Phi(\xi,p)+\int_0^1\{K_1(\xi,\eta,p)+K_2(\xi,\eta,p)\}\Phi(\eta,p)\,\mathrm{d}\eta = \xi. \tag{35}
$$

The kernel functions $K_1(\xi,\eta,p)$ and $K_2(\xi,\eta,p)$ are given by

$$
K_1(\xi,\eta,p) = \frac{2}{\pi}\int_0^\infty\left[F\left(\frac{s}{a},p\right)-1\right]\sin(s\xi)\sin(s\eta)\,\mathrm{d}s \tag{36}
$$

$$
\begin{aligned}
K_2(\xi,\eta,p) = \frac{2}{\pi}\int_0^\infty &\left\{G_1\left(\frac{s}{a},p\right)\sinh(\gamma_3'\xi)\sinh(\gamma_3'\eta)\right. \\
&+G_2\left(\frac{s}{a},p\right)\sinh(\gamma_3'\xi)\sinh(\gamma_4'\eta)+G_3\left(\frac{s}{a},p\right)\sinh(\gamma_4'\xi)\sinh(\gamma_3'\eta) \\
&\left.+G_4\left(\frac{s}{a},p\right)\sinh(\gamma_4'\xi)\sinh(\gamma_4'\eta)\right\}\mathrm{d}s
\end{aligned} \tag{37}
$$

in which the functions $G_i(s,p)$ $(i = 1\text{–}4)$ are given by

$$
\begin{aligned}
G_1(s,p) = &-\frac{(c_{13}\gamma_3+c_{33}\alpha_3)}{\theta_0 c_{11}(\gamma_4^2-\gamma_3^2)\gamma_3}\left[\frac{\delta_1(s,p)}{\gamma_3}(\gamma_3^2u_1-u_2)K_0(\gamma_3 b)\right.\\
&\left.+\{\delta_1(s,p)(\gamma_3^2u_3-u_4)+s\delta_2(s,p)(\gamma_3^2u_5-u_6)\}K_1(\gamma_3 b)\right]\\
G_2(s,p) = &\frac{(c_{13}\gamma_3+c_{33}\alpha_3)}{\theta_0 c_{11}(\gamma_4^2-\gamma_3^2)\gamma_3}\left[\frac{\delta_1(s,p)}{\gamma_4}(\gamma_4^2u_1-u_2)K_0(\gamma_4 b)\right.\\
&\left.+\{\delta_1(s,p)(\gamma_4^2u_3-u_4)+s\delta_2(s,p)(\gamma_4^2u_5-u_6)\}K_1(\gamma_4 b)\right]\\
G_3(s,p) = &-\frac{(c_{13}\gamma_4+c_{33}\alpha_4)}{\theta_0 c_{11}(\gamma_4^2-\gamma_3^2)\gamma_4}\left[\frac{\delta_3(s,p)}{\gamma_3}(\gamma_3^2u_1-u_2)K_0(\gamma_3 b)\right.\\
&\left.+\{\delta_3(s,p)(\gamma_3^2u_3-u_4)+s\delta_4(s,p)(\gamma_3^2u_5-u_6)\}K_1(\gamma_3 b)\right]\\
G_4(s,p) = &\frac{(c_{13}\gamma_4+c_{33}\alpha_4)}{\theta_0 c_{11}(\gamma_4^2-\gamma_3^2)\gamma_4}\left[\frac{\delta_3(s,p)}{\gamma_4}(\gamma_4^2u_1-u_2)K_0(\gamma_4 b)\right.\\
&\left.+\{\delta_3(s,p)(\gamma_4^2u_3-u_4)+s\delta_4(s,p)(\gamma_4^2u_5-u_6)\}K_1(\gamma_4 b)\right]
\end{aligned}
\tag{38}
$$

and $\gamma_3'^2, \gamma_4'^2$ are the two roots of the following quadratic:

$$
c_{11}\gamma^4+\left[(c_{13}^2+2c_{13}-c_{11}c_{33})s^2-(1+c_{11})\frac{a^2p^2}{C_s^2}\right]\gamma^2+\left(s^2+\frac{a^2p^2}{C_s^2}\right)\left(c_{33}s^2+\frac{a^2p^2}{C_s^2}\right)=0. \tag{39}
$$

We note that the kernel function $K_1(\xi,\eta,p)$ is a semi-infinite integral which has a slow rate of convergence. To evaluate the integral in equation (36), we consider the contour integrals:

$$
\begin{aligned}
I_{\Gamma_1} &= \frac{-2}{\pi i(\xi\eta)^{1/2}}\oint_{\Gamma_1} M(w,\gamma_1',\gamma_2')e^{iw\eta}\sin(w\xi)\,d\omega\ (\xi<\eta)\\
I_{\Gamma_2} &= \frac{-2}{\pi i(\xi\eta)^{1/2}}\oint_{\Gamma_2} M(w,\gamma_1',\gamma_2')e^{-iw\eta}\sin(w\xi)\,dw\ (\xi<\eta)
\end{aligned}
\tag{40}
$$

where

$$
M(w,\gamma_1',\gamma_2') = \frac{c_{13}w^2-c_{33}\alpha_1'\gamma_1'-\beta'(c_{13}w^2-c_{33}\alpha_2'\gamma_2')}{(\alpha_1'-\beta'\alpha_2')\theta_0 w}-1 \tag{41}
$$

$$
\begin{aligned}
\gamma_1'(w,p) &= [\tfrac{1}{2}\{-B_1+(B_1^2-4B_2)^{1/2}\}]^{1/2}\\
\gamma_2'(w,p) &= [\tfrac{1}{2}\{-B_1-(B_1^2-4B_2)^{1/2}\}]^{1/2}
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
B_1(w,p) &= \frac{1}{c_{33}}[(c_{13}^2+2c_{13}-c_{11}c_{33})w^2-(1+c_{33})P^2]\\
B_2(w,p) &= \frac{1}{c_{33}}(w^2+P^2)(c_{11}w^2+P^2)
\end{aligned}
\tag{43}
$$

$$\alpha_i'(w,p) = \frac{c_{11}w^2+P^2-\gamma_i'^2}{(1+c_{13})\gamma_i'} \qquad (i=1,2) \tag{44}$$

$$\beta'(w,p) = \frac{\alpha_1'+\gamma_1'}{\alpha_2'+\gamma_2'}, \qquad P = \frac{pa}{C_s}. \tag{45}$$

Now, assuming the relation

$$\left[\frac{(c_{13}^2+2c_{13}-c_{11}c_{33})(1+c_{33})}{c_{33}^2}+\frac{2(1+c_{11})}{c_{33}}\right]^2 - \left[\frac{(c_{13}^2+2c_{13}-c_{11}c_{33})^2}{c_{33}^2}-\frac{4c_{11}}{c_{33}}\right]\left[\frac{(1+c_{33})^2}{c_{33}^2}-\frac{4}{c_{33}}\right] < 0 \tag{46}$$

we find that the roots in the w-plane, denoted by λ_i $(i = 1\sim4)$ of the equation $B_1^2-4B_2 = 0$, are always complex. If we assume the following relations

$$c_{11}c_{33}-c_{13}^2-2c_{13}-1-c_{33} > 0, \qquad c_{13}^2+2c_{13}+c_{11} > 0 \tag{47}$$

the branch points of γ_1' and γ_2' are $\pm Pi$ and $\pm C_{11}^{-1/2}Pi$. Equations (46) and (47) are satisfied for many orthotropic materials. Thus, the contours Γ_1 and Γ_2 are defined as shown in Fig. 2. The integrals in equations (40) satisfy Jordan's lemma on the infinite quarter circles and the integrals on the contours $\Delta\Gamma_1$ and $\Delta\Gamma_2$ along the branch cut become zero. Since $I_{\Gamma_1}+I_{\Gamma_2} = 0$, the kernel $K_1(\xi,\eta,p)$ for $\xi < \eta$ can be finally written as

$$\begin{aligned} K_1(\xi,\eta,P) &= \frac{2}{\pi}\int_0^\infty M(w,\gamma_1',\gamma_2')\sin(w\xi)\sin(w\eta)\,\mathrm{d}w \\ &= -\frac{2}{\pi}P\left[\int_0^{\frac{1}{\sqrt{c_{11}}}} \frac{c_{13}w^2+c_{33}\hat{\alpha}_1\hat{\gamma}_1-\hat{\beta}(c_{13}w^2+c_{33}\hat{\alpha}_2\hat{\gamma}_2)}{(\hat{\alpha}_1-\hat{\beta}\hat{\alpha}_2)\theta_0 w} e^{-P\eta w}\sinh(P\xi w)\,\mathrm{d}w \right. \\ &\left. \quad + \int_{\frac{1}{\sqrt{c_{11}}}}^1 \frac{c_{13}w^2+c_{33}\hat{\alpha}_1\hat{\gamma}_1}{(\hat{\alpha}_1-\hat{\beta}\hat{\alpha}_2)\theta_0 w} e^{-P\eta w}\sinh(P\xi w)\,\mathrm{d}w\right] \quad (\xi<\eta) \end{aligned} \tag{48}$$

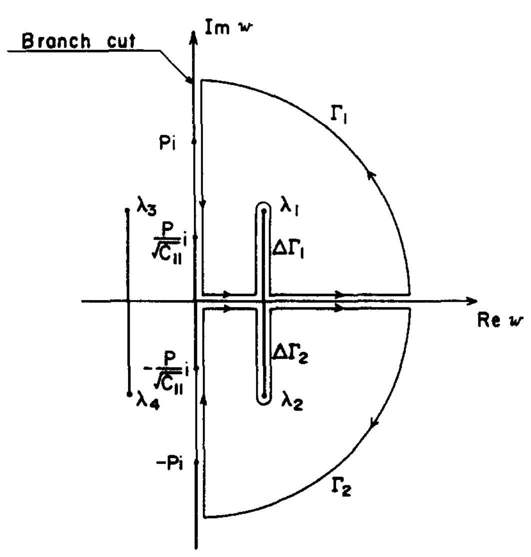

Fig. 2. Contours of integration Γ_1, Γ_2.

in which the functions $\hat{\gamma}_1, \hat{\gamma}_2, \bar{\gamma}_2, \hat{\alpha}_1, \hat{\alpha}_2, \bar{\alpha}_2, \hat{\beta}, \bar{\beta}$ are

$$\begin{aligned} \hat{\gamma}_1(w) &= [\tfrac{1}{2}\{-B'_1+(B_1'^2-4B'_2)^{1/2}\}]^{1/2} \\ \hat{\gamma}_2(w) &= [\tfrac{1}{2}\{-B'_1-(B_1'^2-4B'_2)^{1/2}\}]^{1/2} \\ \bar{\gamma}_2(w) &= [\tfrac{1}{2}\{B'_1+(B_1'^2-4B'_2)^{1/2}\}]^{1/2} \end{aligned} \tag{49}$$

$$\begin{aligned} B'_1(w) &= -\frac{1}{c_{33}}[(c_{13}^2+2c_{13}-c_{11}c_{33})w^2+(1+c_{33})] \\ B'_2(w) &= \frac{1}{c_{33}}(w^2-1)(c_{11}w^2-1) \end{aligned} \tag{50}$$

$$\begin{aligned} \hat{\alpha}_1(w) &= \frac{-c_{11}w^2+1-\hat{\gamma}_1^2}{(1+c_{13})\hat{\gamma}_1} \\ \hat{\alpha}_2(w) &= \frac{-c_{11}w^2+1-\hat{\gamma}_2^2}{(1+c_{13})\hat{\gamma}_2} \\ \bar{\alpha}_2(w) &= \frac{c_{11}w^2-1-\bar{\gamma}_2^2}{(1+c_{13})\bar{\gamma}_2} \end{aligned} \tag{51}$$

$$\begin{aligned} \hat{\beta}(w) &= \frac{\hat{\alpha}_1+\hat{\gamma}_1}{\hat{\alpha}_2+\hat{\gamma}_2} \\ \bar{\beta}(w) &= \frac{\hat{\alpha}_1+\hat{\gamma}_1}{\bar{\alpha}_2+\bar{\gamma}_2}. \end{aligned} \tag{52}$$

The value of the kernel for $\xi > \eta$ is obtained by interchanging ξ and η in equations (48). If we cannot assume the conditions (47), we must develop a new contour or use the original equation (36).

The dynamic stress intensity factor may be determined by obtaining the asymptotic stress $\sigma_z^*(r, 0, P)$ near the crack periphery in the Laplace transform domain and then performing a Laplace inversion. The dynamic singular stress $\sigma_z(r, 0, T)$ may be expressed as

$$\sigma_z(r, 0, T) \sim \frac{k_1(T)}{\sqrt{2(r-a)}} \tag{53}$$

where $T = C_s t/a$ is the nondimensional time and the dynamic stress intensity factor $k_1(T)$ is

$$k_1(T) = \frac{2}{\pi}\sigma_0\sqrt{a}\,\frac{1}{2\pi i}\int_{Br}\frac{\Phi(1,P)}{P}e^{PT}\,dP. \tag{54}$$

Then, a numerical scheme in [3] may be used to evaluate the integral in equation (54).

4. NUMERICAL RESULTS AND DISCUSSIONS

Numerical results have been calculated for the dynamic stress intensity factor. The elastic constants are listed in Table 1[6]. V_f denotes the fiber volume fraction. The relationships between the engineering elastic constants and the properties of the fiber and matrix constituents are given in Appendix B. As $T \to \infty$ and $a/b \to 0$, $k_1(T)$ tends to the static solution $(2/\pi)\sigma_0\sqrt{a}$ for a penny-shaped crack in an infinite solid. The stress intensity factors are normalized by $(2/\pi)\sigma_0\sqrt{a}$.

Table 1. Engineering elastic constants

	E_1(Pa)	E_3(Pa)	μ_{13}(Pa)	ν_{31}	ν_{12}	V_f
	Modulite II graphite–epoxi composite					
Type I	15.3×10^9	158.0×10^9	5.52×10^9	0.34	0.43	0.650
	E-type glass–epoxi composite					
Type II	9.79×10^9	42.3×10^9	3.66×10^9	0.27	0.34	0.565
	Stainless steel–aluminum composite					
Type III	79.76×10^9	85.91×10^9	30.02×10^9	0.33	0.35	0.100

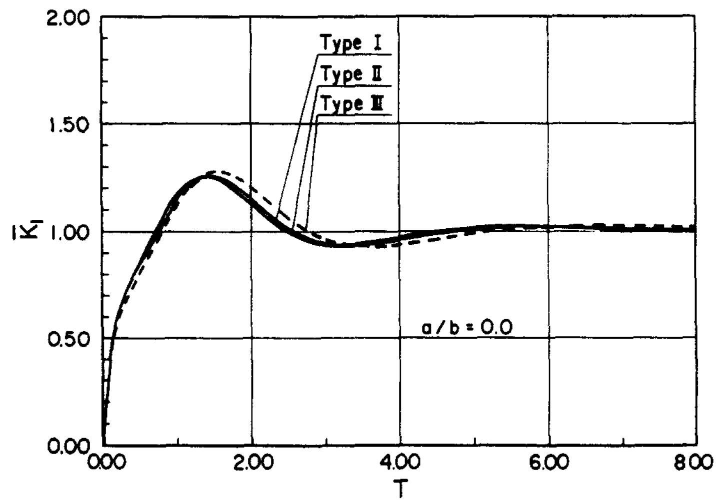

Fig. 3. Dynamic stress intensity factor versus time ($a/b = 0.0$).

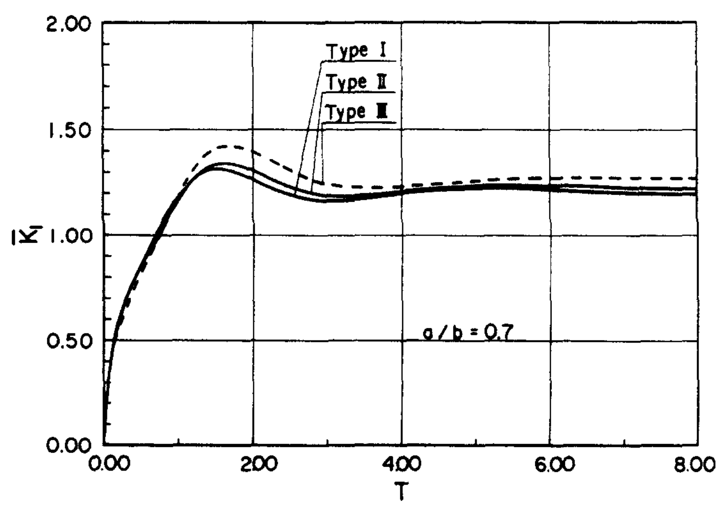

Fig. 4. Dynamic stress intensity factor versus time ($a/b = 0.7$).

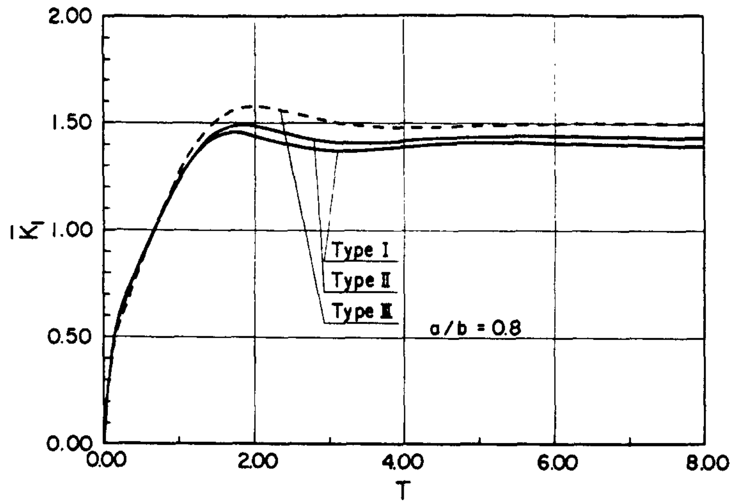

Fig. 5. Dynamic stress intensity factor versus time ($a/b = 0.8$).

In the infinite limit of $a/b = 0$, we derive the results for the dynamic problem of a penny-shaped crack in an infinite transversely isotropic medium. As the results for $a/b = 0$ have not been reported yet, we discuss this problem here, too. Figure 3 exhibits the variation of the normalized dynamic stress intensity factor $\bar{K}_I = k_I(T)/[(2/\pi)\sigma_0\sqrt{a}]$ with the normalized time T for $a/b = 0$ and several composite materials. Theoretically the analysis cannot be applicable to isotropic material. But on taking the values of engineering elastic constants for the type III (stainless steel–aluminum composite), which is nearly isotropic, we have calculated the limiting isotropic case. The result for the type III is in excellent agreement with the result for the isotropic material[5]. The transversely isotropy effect on the peak values of $\bar{K}_I$ are seen to decrease and occur at an earlier time. As $T \to \infty$, $\bar{K}_I$ tends to the static solution $\bar{K}_I = 1.0$ for the penny-shaped crack in an infinite transversely isotropic medium. It is interesting to note that the transverse isotropy has no effect on the static stress intensity factor for $a/b = 0$.

Figures 4 and 5 show the results for the ratios $a/b = 0.7$ and 0.8, respectively. The result for the type III coincides with the result for the isotropic material[5]. The effect of transverse isotropy on the peak values of $\bar{K}_I$ are also observed to decrease and occur at an earlier time. As $T \to \infty$, the dynamic stress intensity factor $\bar{K}_I$ tends to the static solution for the penny-shaped crack in an infinite transversely isotropic cylinder. The transverse isotropy effect is more pronounced with increasing the ratio a/b.

In summary, the dynamic response of a transversely isotropic cylinder with a penny-shaped crack under normal impact is determined in this study. The solution is expressed in terms of the dynamic stress intensity factor. The time dependence of the local stress field is found to depend on the transverse isotropy and the geometrical parameters.

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APPENDIX A

To solve the differential equations (9), we use the integral transforms. Applying the Hankel transform with respect to r to equations (9)[7], the transformed field equations (9) become

$$(D_z^2 - c_{11}s^2 - p^2/C_s^2)\bar{u}_{r1}^* - (1+c_{13})sD_z\bar{u}_{z0}^* = 0$$
$$(1+c_{13})sD_z\bar{u}_{r1}^* + (c_{33}D_z^2 - s^2 - p^2/C_s^2)\bar{u}_{z0}^* = 0 \tag{A1}$$

where

$$\bar{u}_{r1}^*(s,z,p) = \int_0^\infty ru_r^*(r,z,p)J_1(rs)\,\mathrm{d}r$$
$$\bar{u}_{z0}^*(s,z,p) = \int_0^\infty ru_z^*(r,z,p)J_0(rs)\,\mathrm{d}r \tag{A2}$$

$$D_z^2 = \frac{\mathrm{d}^2}{\mathrm{d}z^2}. \tag{A3}$$

From equations (A1) we get

$$[c_{33}D_z^4 + \{(c_{13}^2 + 2c_{13} - c_{11}c_{33})s^2 - (1+c_{33})(p/C_s)^2\}D_z^2 + \{s^2 + (p/C_s)^2\}\{c_{11}s^2 + (p/C_s)^2\}]\bar{u}_{r1}^* = 0. \tag{A4}$$

A proper solution to equation (A4) which vanishes for large z is

$$\bar{u}_{r1}^* = \frac{1}{s}\{A_1(s,p)e^{-\gamma_1 z} + A_2(s,p)e^{-\gamma_2 z}\}. \tag{A5}$$

Substituting this expression into the first equation of (A1) and integrating over z, we find that

$$\bar{u}_{z0}^* = \frac{1}{s^2}\{\alpha_1 A_1(s,p)e^{-\gamma_1 z} + \alpha_2 A_2(s,p)e^{-\gamma_2 z}\}. \tag{A6}$$

Similarly, applying the Fourier transform with respect to z to equations (9)[7], we arrive at the equations

$$\{c_{11}D_r^2 - s^2 - (p/C_s)^2\}\bar{u}_r^* - is(1+c_{13})\bar{u}_{z,r}^* = 0$$
$$-is(1+c_{13})D_r^2\bar{u}_r^* + \{D_r^2 - c_{33}s^2 - (p/C_s)^2\}\bar{u}_{z,r}^* = 0 \tag{A7}$$

where

$$\bar{u}_r^*(r,s,p) = \int_{-\infty}^\infty u_r^*(r,z,p)e^{isz}\,\mathrm{d}z$$
$$\bar{u}_z^*(r,s,p) = \int_{-\infty}^\infty u_z^*(r,z,p)e^{isz}\,\mathrm{d}z \tag{A8}$$

$$D_r^2 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}. \tag{A9}$$

From equations (A7) we also get

$$[c_{11}D_r^4 + \{(c_{13}^2 + 2c_{13} - c_{11}c_{33})s^2 - (1+c_{11})(p/C_s)^2\}D_r^2 + \{s^2 + (p/C_s)^2\}\{c_{33}s^2 + (p/C_s)^2\}]\bar{u}_r^* = 0. \tag{A10}$$

Suitable forms of $\bar{u}_r^*$ and $\bar{u}_z^*$ are taken as

$$\bar{u}_r^*(r,s,p) = \pi\{A_3(s,p)I_1(\gamma_3 r) + A_4(s,p)I_1(\gamma_4 r)\}$$
$$\bar{u}_z^*(r,s,p) = \frac{\pi i}{s}\{\alpha_3 A_3(s,p)I_0(\gamma_3 r) + \alpha_4 A_4(s,p)I_0(\gamma_4 r)\}. \tag{A11}$$

Table 2. Material properties of fibers and matrices

Fiber	E_f(Pa)	μ_f(Pa)	ν_f
Modulite II graphite	241.5×10^9	92.9×10^9	0.30
E-type glass	72.5×10^9	30.4×10^9	0.20
Stainless steel	207.0×10^9	79.6×10^9	0.30
Matrix	E_m(Pa)	μ_m(Pa)	ν_m
Epoxi	3.11×10^9	1.17×10^9	0.35
Aluminum	72.5×10^9	27.2×10^9	0.33

Inverting equations (A5), (A6) by means of the Hankel inversion theorem and equations (A11) by means of the Fourier inversion theorem[7] and using the standard superposition technique, we obtain expressions (10).

APPENDIX B

The relationships between the elastic constants shown in Table 1 and the properties of the fiber and matrix are given below.

$$
\begin{aligned}
E_1 &= \left[\frac{E_f+E_m+(E_f-E_m)V_f}{E_f+E_m-(E_f-E_m)V_f}\right]E_m \\
E_3 &= E_f V_f + E_m V_m \\
\nu_{31} &= \nu_f V_f + \nu_m V_m \\
\nu_{12} &= \nu_f V_f + \left[\frac{1+\nu_m-\nu_{31}E_m/E_3}{1-\nu_m^2+\nu_m\nu_{31}E_m/E_3}\right]\nu_m V_m \\
\mu_{13} &= \left[\frac{\mu_f+\mu_m+(\mu_f-\mu_m)V_f}{\mu_f+\mu_m-(\mu_f-\mu_m)V_f}\right]\mu_m
\end{aligned}
\tag{B1}
$$

where V_f, V_m $(=1-V_f)$ are volume fractions, E_f, E_m are Young's moduli, ν_f, ν_m are Poisson ratios, μ_f, μ_m are shear moduli and the subscripts f and m denote the properties of the fiber and matrix. The properties of the fiber and matrix are shown in Table 2.